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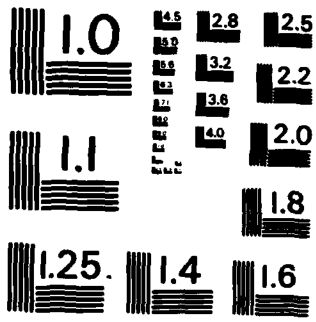

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A note on the prediction error for small time lags into the future

by

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A Note on the Prediction Error for Small Time Lags into the Future

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Abstract

Explicit expressions are derived for derivatives at zero lag of the mean square prediction error of a class of random processes which includes the rational case. A simple random process is demonstrated which is not in the class. A sufficient condition is given to determine whether a random process is in the class.

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Introduction

Finding the minimum variance linear predictor of a continuous time wide sense stationary random process is a classical problem in the electrical engineering and mathematics literature. Wiener [1] solved the problem of prediction given that the infinite past of the process is observed. Krein [2] solved the problem in principle for the much harder case of observing only a finite portion of the entire past. The problem is that the prediction formula is still extremely difficult to calculate in a practical sense. Rozanov [3] gives formulae for the rational power spectrum case but these still require solving complex differential equations to obtain certain constants. Given that the predictors may be difficult to calculate one may desire to have recourse to suboptimal ad-hoc schemes. If so, it is of crucial importance to be able to calculate the mean squared error performance of such a suboptimal predictor and compare it to the performance of the optimal one.

More recently Cuzick [4] gives a nice overview of the problem and derives an upper bound to the prediction error which approaches zero at the correct rate as the "lag" or time interval into the future approaches zero. These bounds however have a unknown constant appearing in them which hampers (as the author states) the practical utility of the results.

In this note we will give approximate expressions for the mean squared prediction error for small extrapolations into the future for a class of random processes which includes the rational case. Much of our intuition is based upon how filtering and prediction works for the rational case. As an interesting sidelight it is demonstrated that the predictors (and their error) can depart very widely from those that would be expected based upon a rational approximation.

Notation

(1) $\{x(t) \; -\infty \leq -T \leq t \leq 0\}$ is an observation of a wide sense stationary nondeterministic zero mean random process.

(2) $x^{(k)}(t)$ is the kth mean square derivative of $x(t)$ (if it exists).

(3) $\hat{x}_T(\tau)$ be the minimum mean squared error predictor of $x(\tau)$, $\tau \geq 0$.

(4) $E\{(x(\tau) - \hat{x}_T(\tau))^2\} \triangleq mse_T(\tau)$.

(5) $F'(\lambda)$ is the power spectrum of the random process $x(t)$. $F^{-1}\{F'(\lambda)\} = R(t)$ the autocorrelation function the random process where $F\{\ \}$ and $F^{-1}\{\ \}$ denote the Fourier and inverse Fourier Transform operators respectively.

(6) $R(t) = f(t) * f(-t)$ where $f(t)$ is a causal square integrable function which is guaranteed to exist by the Paley-Wiener Theorem [5].

Development

Suppose $F'(\lambda)$ is rational and $F'(\lambda) \neq 0$, then $F\{f(t)\}$ is also rational.

$$F'(\lambda) = \left| \frac{\sum_{j=0}^{\alpha} A'_j (i\lambda)^j}{\sum_{j=0}^{\beta} B'_j (i\lambda)^j} \right|^2$$

where A'_j and B'_j are real, and $\beta \geq \alpha+1$.

$$F\{f(t)\} = \frac{\sum_{j=0}^{\alpha} A'_j (i\lambda)^j}{\sum_{j=0}^{\beta} B'_j (i\lambda)^j}$$

Let k be such that $\beta \geq \alpha+k+1$.

Then the Laplace Transform of the kth derivative of f(t) is given by,

$$\mathcal{L}\{f^{(k)}(t)\} = \frac{(s^k) \sum_{j=0}^{\alpha} A_j'(\frac{s}{2\pi})^j}{\sum_{\gamma=0}^{\beta} B_j'(\frac{s}{2\pi})^j} .$$

By the Abelian Theorem for Laplace Transforms, we have

$$f^{(k)}(0) = \lim_{s\to\infty} \mathcal{L}\{f^{(k)}(t)\}s$$

$$= 0 \qquad \beta > \alpha+k+1$$

$$= \frac{A_\alpha'}{B_\beta'} (2\pi)^{k+1} \qquad \beta = \alpha+k+1$$

However from [6, pg. 544] we have

$$R^{(k)}(0^+) = 0 \qquad 2\beta \geq 2\alpha+k+2$$

$$= +\frac{1}{2} \left|\frac{A_\alpha'}{B_\beta'}\right|^2 (2\pi i)^{2\beta-2\alpha} \qquad 2\beta = 2\alpha+k+1$$

Hence

$$f^{(k)}(0)^2 = 2R^{(2k+1)}(0^+)(-1)^{k+1} \tag{1}$$

$$(\text{for } \beta-\alpha-1 \geq k)$$

A random process of the above type has exactly $\beta-\alpha-1$ derivatives. Therefore we can consider a suboptimal predictor $\hat{x}(\tau)$, of $x(\tau)$ as

$$\hat{x}(\tau) \triangleq \sum_{k=0}^{\beta-\alpha-1} \frac{\tau^k}{k!} x^{(k)}(0) .$$

We define

$$\widetilde{mse}(\tau) \triangleq E\{(x(\tau) - \hat{x}(\tau))^2\}$$

It is straightforward (but extremely tedious) to verify that

$$\left.\frac{\partial^{(k)}\widetilde{mse}(\tau)}{\partial\tau^k}\right|_{\tau=0^+} = 2\binom{k-1}{\frac{k-1}{2}} R^{(k)}(0^+)(-1)^{\frac{k+1}{2}} \quad k \leq 2(\beta-\alpha)-1$$

A well known formula [5,6] for the optimum prediction when $T = \infty$ is given by

$$mse_\infty(\tau) = E\{(x(\tau) - \hat{x}_\infty(\tau))^2\} = \int_0^\tau f^2(t)dt$$

Again it is straightforward but tedious to verify that

$$\left.\frac{\partial^k mse_\infty(\tau)}{\partial\tau^k}\right|_{\tau=0^+} = \binom{k-1}{\frac{k-1}{2}}(f^{(\frac{k-1}{2})}(0^+))^2 \quad k \leq 2(\beta-\alpha)-1$$

$$= \left.\frac{\partial^{(k)}\widetilde{mse}(\tau)}{\partial\tau^k}\right|_{\tau=0^+}$$

(we emphasize again that the above is zero unless $k = 2(\beta-\alpha)-1$). Hence since $\widetilde{mse}(\tau)$ represents the error of a suboptimal estimator which depends only upon the value of the random process and its derivatives at zero we must have

$$K! \frac{\widetilde{mse}(\tau)}{\tau^k} \geq K! \frac{mse_T(\tau)}{\tau^k} \geq K! \frac{mse_\infty(\tau)}{\tau^k}$$

where $k \leq 2(\beta-\alpha)-1$. Taking the limit as $\tau \to 0^+$ we have our theorem:

Theorem 1

Suppose the random process $x(t)$ has a rational power spectrum $F'(\lambda)$ such that $F'(\lambda) \neq 0$.

$$\left.\frac{\partial^k mse_T(\tau)}{\partial\tau^k}\right|_{\tau=0^+} = 2\binom{k-1}{\frac{k-1}{2}} R^{(k)}(0^+)(-1)^{\frac{k+1}{2}} \qquad k \leq 2(\beta-\alpha)-1$$

where $R(\tau) = F^{-1}\{F'(\lambda)\}$.

Remark 1

The above development proves the existence of the partial derivative and gives its value at $\tau = 0^+$. Note that for all values of K smaller than $2(\beta-\alpha)-1$ the value is zero.

Remark 2

By the nature of the proof we see that for small prediction "lags" the simple Taylor series predictor performs just about as well as the more complex Wiener predictor for processes with rational power spectra.

One may now hope that a more general class of random processes will behave in the same manner. Since the rationals are dense in the class of all spectra we could hope that all random processes would behave like this. This is not the case as shown by the following counterexample:

Counterexample

Suppose $F'(\lambda) = \frac{\sin^2(\frac{\lambda}{2})}{\lambda^2}$ and $\frac{T}{2} = N$ a positive integer. Then from [5] we have

$$mse_T(\tau) = \frac{2N+2}{8(2N+1)}\tau + \frac{(N+1-\tau)\tau}{4(2N+1)} \qquad 0 \le \tau < 1/2$$

$$\left.\frac{\partial mse_T(\tau)}{\partial \tau}\right|_{\tau=0^+} = \frac{2N+2}{(2N+1)8} + \frac{N+1}{(2N+1)4}$$

$$= \frac{N+1}{(2N+1)2}$$

Suppose we use $\hat{\hat{x}}(\tau) = \frac{R(\lambda)}{R(0)} x(0)$ (the best Taylor Series predictor). Then since

$$R(\lambda) = \begin{cases} \frac{1}{4} - \frac{1}{4}\lambda & \text{for} \quad 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$

$$\widetilde{mse}(\tau) = \frac{1}{4} - \frac{(\frac{1}{4} - \frac{1}{4}\lambda)^2}{1/4}$$

$$\left.\frac{\partial \widetilde{mse}(\tau)}{\partial \tau}\right|_{\tau=0^+} = \frac{1}{2}$$

Remark 1

Obviously these two expressions are unequal with the optimal predictor error derivative strictly less than 1/2 for all integer values of N and approaching 1/4 as N approaches ∞.

Remark 2

In contrast to the rational case the partial of the error depends upon the length of the observation T we are given.

However if we impose some technical conditions we can demonstrate an entire class of power spectra that will act as in the rational case.

Theorem 2

Suppose f(t) has the following representation

$$f(t) = f(0) + \int_0^t f'(\alpha)d\alpha$$

where $f'(\alpha)$ is square integrable i.e. f(t) is absolutely continuous and its derivative is a square integrable function. Then $R'(0^+) = \frac{-f^2(0)}{2}$.

Proof

For $\tau > 0$ we have

$$R(\tau) = \int_\tau^\infty f(\alpha)f(\alpha-\tau)d\alpha$$

$$= \int_\tau^\infty f(\alpha)[f(\alpha) - \int_0^\tau f'(\alpha-s)ds]d\alpha$$

(by absolute continuity of f)

$$= \int_\tau^\infty f^2(\alpha)d\alpha - \int_0^\tau \int_0^\infty f(\alpha)f'(\alpha-s)d\alpha ds$$

$$+ \int_0^\tau \int_0^\tau f(\alpha)f'(\alpha-s)d\alpha ds$$

(integration interchange justified by invoking Tonelli's Theorem on $|f(\alpha)f'(\alpha-s)|$ to get integrability on cross measure then Fubini's Theorem allows interchange). Now consider the last term divided by τ i.e.

$$\frac{1}{\tau}\int_0^\tau \int_0^\tau f(\alpha)f'(\alpha-s)d\alpha ds$$

$$\leq \frac{1}{\tau} \int_0^\tau \left(\int_0^\tau f^2(\alpha)d\alpha\right)^{1/2}\left(\int_0^\tau f'^2(\alpha-s)d\alpha\right)^{1/2} ds$$

(by Schwartz inequality)

$$\leq \frac{1}{\tau} \int_0^\tau K\left(\int_0^\tau f^2(\alpha)d\alpha\right)^{1/2} ds$$

$$= \frac{1}{\tau}\, \tau\, K(mse_\infty(\tau))^{1/2} = mse_\tau^{1/2}\, K \xrightarrow[\tau\to 0^+]{} 0$$

Therefore we may now safely invoke the fundamental theorem of the calculus and find (for $\tau > 0$)

$$R'(\tau) = -f^2(\tau) - \int_0^\infty f(\alpha)f'(\alpha-\tau)d\alpha + 0(\tau)$$

where we have terms that are approaching zero as τ approaches zero by the above agrument. Utilizing the fact that f' is square integrable we may approximate it uniformly close by a continuous function which implies we can take limit as τ goes to zero through positive values to obtain

$$R'(0^+) = -f^2(0) - \int_0^\infty f(\alpha)f'(\alpha)d\alpha$$

$$= -f^2(0) - \left.\frac{f^2(\alpha)}{2}\right|_0^\infty$$

$$= -f^2(0) + \frac{f^2(0)}{2} = -\frac{f^2(0)}{2}$$

Note the next to last line follows since

$$\int_0^\infty |f(x)|\ |f'(x)|dx = \lim_{T\to\infty} \int_0^T |f(x)|\ |f'(x)|dx$$

$$\leq \|f\|\ \|f'\| = M < \infty$$

which implies the following limit exists

$$\lim_{T\to\infty} \int_0^T f(x)f'(x)dx = M' < \infty$$

because absolute summability implies summability. Therefore $\lim_{T\to\infty} f(T)^2$ exists and hence must equal zero.

Q.E.D.

Remark 1

The theorem statement corresponds to Eq. (1) with k=0. We could obtain a statement for arbitrary k by making the same assumptions on $f^{(k-1)}(t)$ that we do for f(t). Although the proof is messier, it is essentially the same.

Remark 2

The counterexample fails the theorem requirements since

$$f(t) = \frac{1}{2} \qquad 0 \leq t \leq 1$$

we would have to have $f'(t) = -\frac{1}{2}\,\delta(t-1)$ and this candidate for the derivative of course is not square integrable.

Remark 3

One can now generate entire families of power spectra that behave as the rational spectra do. For example take

$$
\begin{aligned}
f(t) &= 1 && 0 \le t \le 1 \\
&= e^{-(t-1)} && t \ge 1 \\
&= 0 && t < 0 \\
f'(t) &= 0 && 0 \le t \le 1 \\
&= -e^{-(t-1)} && t > 1 \\
&= 0 && t < 0
\end{aligned}
$$

and it is square integrable.

$$R_x(\tau) = 2-\tau - \frac{e^{-\tau}}{2} \qquad 0 \le \tau \le 1$$

$$\left.\frac{\partial R_x(\tau)}{\partial \tau}\right|_{\tau=0} = -\frac{1}{2} = -\frac{f(0)^2}{2}$$

Remark 4

If $f \epsilon L^2$ and $f' \epsilon L^2$ a well known property of convolutions states that $f*f'$ is not only continuous but uniformly continuous. Hence this implies the derivative of the autocorrelation function must exist at every point (see proof of theorem 2) and be continuous (except possibly at the origin). This condition would usually be easier to check on a particular autorcorrelation function than if $f, f' \epsilon L^2$. Our counterexample with the triangular autocorrelation of course fails this test.

Conclusions

Basically this note reports on an interesting phenomenon dealing with the prediction error for small lags. For the rational power spectrum case a closed form expression is derived for the derivative with respect to the lag of the mean square prediction error. It is shown that, although it appears a large class of "nice" power spectra do behave like the rationals, there are also simple ones that don't, in particular the triangular autocorrelation function. For these "nice" random processes simple Taylor Series type predictors perform optimally well for small lags into the future.

The problem of whether a particular power spectrum is "nice" or not remains an open problem. A sufficient condition given in terms of smoothness attributes on the functions in the Wiener spectral decomposition has been given but its utility is limited due to the difficulty of performing the decomposition and verifying the conditions. We are able to remark though that if the autocorrelation function isn't everywhere differentiable (except possibly at zero) then it will not be able to meet our smoothness conditions.

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